ANALYTIC SEMIGROUPS AND APPLICATIONS

KELLER VANDEBOGERT

1. Introduction

Consider a Banach space [1] X and let $f : D \to X$ and $u : G \to X$, where D and G are real intervals. A is a bounded or unbounded linear operator whose domain is densely defined in X. We then consider:

$$\frac{du}{dt} + A(t)u = f(t)$$

The above is referred to as an evolution equation. We can impose an initial condition, say, $u(0) = x$, and this evolution becomes what is known as *The Abstract Cauchy Problem.* In general, there are different methods of solution for these types of problems, but no single method always seems to work. This is where semigroups can be used.

Consider an operator, $T(t)$, which can be thought of as an "evolution" operator. $T(t)$ applied to $u(t_0)$ will have the following effect:

$$T(t)u(t_0) = u(t_0 + t)$$

This is an interesting consideration, and when the initial condition is taken into account, immediately leads to the following:

$$u(t) = T(t)x$$

We can then reformulate our original evolution equation in a very obvious way. We now consider the following: intuitively, if we were to

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[1] A Banach space X is defined as a normed space in which all Cauchy sequences converge to a point $x \in X$. (A *complete* normed space)

apply our evolution operator in succession, it would just have the effect of shifting our time twice. More precisely,

$$T(s)T(t)x = u(t+s) = T(t+s)x$$

We have just plunged into the world of semigroups. Put very simply, a semigroup is merely a group without the inverse or identity property. In the language of what we have just presented, we have:

$$T(s)[T(t)x] = [T(s)T(t)]x$$

and

$$T(s)T(t) = T(s+t)$$

This leads us to considerations of our original evolution equation and to consider how A fits into all of this. In fact, A will be the deciding factor in terms of the nature of our solution. It can be referred to as the infinitesimal generator, and we will explore some of the desired properties of A in order to have a well behaved solution, and to in fact solve the nonhomogeneous Abstract Cauchy Problem.

2. Semigroups: Strongly Continuous and Analytic

The abstract definition of a semigroup has already been introduced. Namely, we have a group whose elements need not be invertible, nor contain an identity element. For our purposes, this is a little bit too general to be of use, and so it is desirable to consider other properties that may naturally occur in the setting of evolution equations. This motivates the definition of the *strongly continuous*, or C_0 *semigroup.*

Definition 2.1. A collection $\{T(t)\}$, where $t \in [0, \infty)$, of bounded linear operators in X is called a C_0 semigroup if:

(1) $T(s+t) = T(s)T(t)$ for all $s, t > 0$
(2) $T(0) = I$ (the identity operator)
(3) For each $x \in X$, $T(t)x$ is continuous in t on $[0, \infty)$

Note that X is assumed to be a Banach Space. Also, if our third condition can be strengthened to the case where we have continuity in the uniform operator (norm) topology, then we have a uniformly continuous semigroup. We now want to introduce another notion: that of a generator for the semigroup. However, we do not have generators in the traditional sense of group theory. Indeed, we have something a little more exotic.

Definition 2.2. Let $h > 0$. Then, A is called an *infinitesimal generator* of the semigroup $\{T(t)\}$ if

$$Ax = \lim_{h \to 0} \frac{T(h)x - x}{h}$$

This definition is actually quite natural if you just consider the definition of the derivative of $T(t)$, and then utilize the properties of semigroups. This consideration leads us to our first lemma.

Lemma 2.3. *Let $\{T(t)\}$ be a strongly continuous semigroup with the infinitesimal generator A. Then, for any $x \in D_A$ (the domain of A):*

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax$$

Proof. Let $t, h > 0$. By definition of derivative:

$$\begin{aligned} \frac{d}{dt}T(t)x &= \lim_{h\to 0}\frac{T(t+h)x - T(t)x}{h} \\ &= \lim_{h\to 0}\frac{T(t)T(h)x - T(t)x}{h} \\ &= T(t)\lim_{h\to 0}\frac{T(h)x - x}{h} \\ &= T(t)Ax \end{aligned} \tag{2.1}$$

Also,

$$\begin{aligned} \lim_{h\to 0}\frac{T(t)x - T(t-h)x}{h} &= \lim_{h\to 0}T(t-h)\frac{T(h)x - x}{h} \\ &= T(t)Ax \end{aligned}$$

Commutativity follows immediately by the commutativity of addition. Namely,

$$\begin{aligned} \lim_{h\to 0}\frac{T(h+t)x - T(t)x}{h} &= \lim_{h\to 0}\frac{T(h) - I}{h}T(t)x \\ &= AT(t)x \end{aligned} \tag{2.2}$$

As desired.

□

Note that we proved equality for h tending from both the left and the right, since we are only working in the strong operator topology. We now want to introduce the concept of an analytic semigroup. As we shall see, analytic semigroups are a restriction on the set of C_0 semigroups, and this class of semigroups in fact provides better regularity of solutions for PDE's.

Definition 2.4. Let $\{T(t)\}$ be a C_0 semigroup on a Banach Space X with infinitesimal generator A. Then, $\{T(t)\}$ is said to be an analytic semigroup if:

(1) For some $\phi \in (0, \pi/2)$, $T(t)$ can be extended to Δ_ϕ, where:

$$\Delta_\phi = \{0\} \cup \{t \in \mathbb{C} : |\arg(t)| < \phi\}$$

(2) For all $t \in \Delta_\phi - \{0\}$, we have that $T(t)$ is analytic in t in the uniform operator topology.

In a less formal manner, analytic semigroups are C_0 semigroups in which each $T(t)$ has an analytic continuation to the sector Δ_ϕ, in which the local power series representation of $T(t)$ converges in norm. As we shall see, this type of semigroup has a natural association to the Abstract Cauchy Problem.

3. The Abstract Cauchy Problem

As mentioned in the introduction, the Abstract Cauchy Problem is to find a function $u(t)$ such that:

$$\frac{du}{dt} + A(t)u = f(t) \tag{3.1}$$

where $u(0) = x$, and A can be an either bounded or unbounded linear operator.

If we examined the homogeneous case of (3.1), it is possible to pose a rather naive solution. Namely, if

$$\frac{du}{dt} + Au = 0$$

Then,

$$u(t) = e^{-At}x$$

This solution of course seems completely ridiculous. What does it even mean to exponentiate an operator? Surprisingly, this approach can be shown to be well defined.

From Section 1, we know that we can repose this problem in the language of our "evolution" operator. We have:

$$\frac{d}{dt}(T(t)x) + A(t)T(t)x = 0 \tag{3.2}$$

With (3.2) and Definition 2.2, we see that if $-A$ is an infinitesimal generator of $\{T(t)\}$, then, symbolically, we have a solution. Comparing this with our "naive" solution, this implies that $T(t)x = e^{-At}x$.

Interestingly, all of the operations line up. Assuming that -A is our infinitesimal generator, we see:

(1) $T(s)T(t) = e^{-As}e^{-At} = e^{-A(s+t)} = T(s+t)$
(2) $\frac{d}{dt}(T(t)x) = \frac{d}{dt}(e^{-At}x) = -AT(t)x = -Ae^{-At}x$

And other properties are readily verified. Also, we note that the above properties are merely based off of the assumption that e^{-At} will act the same as the regular exponential function, which we intend to prove. It is extremely important to note that the operation between the above expressions is not multiplication. This is an arbitrary operation, and because of this, the above relations are not trivial.

To make sense of this, we will have to look at e^{-At} in a different way.

4. Characterization of Infinitesimal Generators

From elementary definitions of the exponential function, we have 3 ways to define e^{At}.

(1) $e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$
(2) $e^{At} = \lim_{n\to\infty} \left(1 + \frac{tA}{n}\right)^n$
(3) $e^{At} = \mathcal{L}^{-1}((\lambda I - A)^{-1})$

Where $\mathcal{L}^{-1}(.)$ is the inverse Laplace Transform. From our first possible definition, we see

Theorem 4.1. *Let $A : X \to X$ be a bounded linear operator. Then,*

$$T = \left\{ T(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right\}$$

is a uniformly continuous semigroup.

Proof. Firstly, $||A|| < \infty$ since our operator is bounded. We first show that $T(s)T(t) = T(s+t)$.

$$T(s)T(t) = e^{At}e^{As} = \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} \sum_{j=0}^{\infty} \frac{(sA)^j}{j!} = \sum_{n=0}^{\infty} \frac{((s+t)A)^n}{n!}$$

This of course holds by the properties of the exponential. Also, setting $t = 0$, it is obvious that the only term in our summation is I, the identity operator.

Finally, to show this is a uniformly continuous semigroup, we need to show that $T(t) \to I$ as $t \to 0^+$ in norm. We see:

$$||T(t) - I|| = || \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} || \leqslant \sum_{n=1}^{\infty} \frac{(t||A||)^n}{n!} = e^{t||A||} - 1$$

Letting $t \to 0^+$, we see that the norm tends to 0.

□

Thus, for bounded operators we see that this is in fact well defined. However, not all operators are bounded. Here, we will state the theorem of Hille-Yosida without proof, since it is well beyond the scope of the paper(see [2]). However, this theorem gives a very broad characterization of linear operators which are infinitesimal operators of C_0 semigroups.

Theorem 4.2 (Hille-Yosida Theorem)**.** *A necessary and sufficient condition that a closed linear operator A with dense domain D_A be the infinitesimal generator of a C_0 semigroup is that there exist real numbers M and ω such that for every real $\lambda > \omega$, $\lambda \in \rho(A)$, and:*

$$||R(\lambda; A)^n|| \leqslant \frac{M}{(\lambda - \omega)^n}$$

We note that $R(\lambda; A) = (\lambda I - A)^{-1}$, and $\rho(A)$ is the resolvent set of A. Now, moving back toward the Abstract Cauchy Problem, we want to see what kind of semigroup $-A$ generates. To do this, we can make some assumptions on the operator A. Recall that a closed operator $A : D_A \to Y$ is one such that if for any sequence $x_n \to x$ in D_A we have that $Ax_n \to y$, then $x \in D_A$ and $Ax = y$.

Definition 4.3. We say the operator A is of type (ϕ, M) if:

(1) A is a closed operator with D_A dense in X
(2) $\{\lambda : \lambda \neq 0, \pi/2 - \phi < \arg(\lambda) < 3\pi/2 + \phi\} \subset \rho(A)$, and:

$$||R(\lambda; A)|| \leqslant \frac{M}{|\lambda|}$$

Theorem 4.4. *If A is of type (ϕ, M), then $-A$ generates an analytic semigroup $\{T(t)\}$.*

Note in the below proof we will use $T(t)$ and e^{-At} interchangeably.

Proof. Define

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda; -A) d\lambda \tag{4.1}$$

where Γ is a contour in the complex plane. Let Γ be defined as two segments: $\{re^{i\theta_i} : r \geqslant 1\}$, for $i = 1, 2$. Additionally, $\pi/2 < \theta_1 < \pi/2 + \phi$, and $3\pi/2 - \phi < \theta_2 < 3\pi/2$. Connect these two curves by the portion of the unit circle such that $\theta_1 \leqslant \theta \leqslant \theta_2$. We then orient Γ such that for $\lambda = re^{i\theta_1}$, $d\lambda = ie^{i\theta_1}dr$.

With this contour it can be see that the integral above is well defined. This integral converges absolutely and is a bounded operator. We then let f be any bounded linear functional in X. Then, we form a new contour Γ' by translating Γ to the right a small distance. Then, we have:

$$e^{-sA} = \frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda' s} R(\lambda'; -A) d\lambda'$$

Consider:

$$f\left(\frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda' s} R(\lambda'; -A) d\lambda'\right) = \frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda' s} f\big(R(\lambda'; -A)\big) d\lambda'$$

Since Γ was translated to Γ' without passing any additional singularities, we have by Cauchy's Theorem:

$$\frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda' s} f\big(R(\lambda'; -A)\big) d\lambda' = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda s} f\big(R(\lambda; -A)\big) d\lambda$$

We now consider:

$$e^{-tA}e^{-sA} = \frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda t + \lambda' s} R(\lambda; -A) R(\lambda'; -A) d\lambda d\lambda'$$

Employing the resolvent equation, this becomes:

$$\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda t + \lambda' s} \frac{1}{\lambda' - \lambda}\Big[R(\lambda; -A) - R(\lambda'; -A)\Big] d\lambda d\lambda'$$

Since Γ lies to the left of Γ', $\lambda \neq \lambda'$ when integrating over Γ. With this and Fubini's theorem, we can reduce the integrand.

$$\frac{1}{(2\pi i)^2}\int_{\Gamma}\int_{\Gamma'} e^{\lambda t + \lambda' s} \frac{1}{\lambda' - \lambda}\Big[R(\lambda; -A) - R(\lambda'; -A)\Big] d\lambda d\lambda'$$

$$= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e^{\lambda t+\lambda' s}\frac{1}{\lambda'-\lambda}R(\lambda;-A)d\lambda d\lambda' - \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e^{\lambda t+\lambda' s}\frac{1}{\lambda'-\lambda}R(\lambda';-A)d\lambda d\lambda'$$

The second term in the above expression is 0 with the use of Cauchy's theorem. The first expression simplifies to:

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t+s)}R(\lambda;-A)d\lambda = e^{-(t+s)A} = e^{-tA}e^{-sA}$$

And we have proved the semigroup property. We still need to show that this is a C_0 semigroup satisfying the additional conditions of Definition 2.4.

Let $\epsilon > 0$. If we can show that $T(t)$ has an analytic continuation along any curve in the sector $\Delta_{\phi-\epsilon}$, then $T(t)$ has an analytic continuation in all of Δ_ϕ. Thus, we consider a curve Γ in the sector $\Delta_{\phi-\epsilon}$. Since A is of type (ϕ, M),

$$||R(\frac{\lambda}{|t|};-A)|| \leqslant \frac{C|t|}{|\lambda|}$$

Where $C = C(\epsilon)$. Letting $\lambda' = |t|\lambda$, we can scale our contour such that $\Gamma' = |t|\Gamma$. Then, since Γ' will not contain any new singularities:

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda;-A)d\lambda = \frac{1}{2\pi i}\int_\Gamma e^{\lambda' \arg(t)}R(\lambda'/|t|;-A)d\lambda'/|t|$$

Then,

$$||e^{-tA}|| \leqslant C\int_\Gamma |e^{\lambda' \arg(t)}||d\lambda'|/|\lambda'| \leqslant C \tag{4.2}$$

Also, since A is a closed operator,

$$Ae^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda' \arg(t)}AR(\lambda'/|t|;-A)d\lambda'/|t|$$

Rewrite this as:

$$\frac{1}{2\pi i}\left[\int_\Gamma e^{\lambda' \arg(t)}\Big(\lambda'/|t|I+A\Big)R(\lambda'/|t|;-A)d\lambda'/|t|-\int_\Gamma e^{\lambda' \arg(t)}\lambda'/|t|R(\lambda'/|t|;-A)d\lambda'/|t|\right]$$

The first integral is 0 by Cauchy's Theorem. We can then bound Ae^{-tA} by using (4.2).

$$(4.3) \quad ||Ae^{-tA}|| \leqslant \frac{1}{|t|}\int_\Gamma e^{\lambda' \arg(t)}|\lambda'| * ||R(\lambda'/|t|;-A)|| * |d\lambda'|/|t| \leqslant \frac{C}{|t|}$$

With this, we know that $T(t)$ and $AT(t)$ are bounded operators. If we can show that $\frac{dT(t)x}{dt} = -AT(t)x$, then this means that $T(t)$ can be locally represented by a power series in the uniform operator topology. By the principle of analytic continuation, it will then be possible to extend $T(t)$ to all of Δ_ϕ. Using (4.1):

$$\frac{d}{dt}T(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda;-A)d\lambda$$

Rewrite $\lambda R(\lambda;-A)$ as $I - AR(\lambda;-A)$:

$$\frac{1}{2\pi i}\left[\int_\Gamma e^{\lambda t}d\lambda - A\int_\Gamma e^{\lambda t}R(\lambda;-A)d\lambda\right]$$

The first integral vanishes since the integrand is holomorphic, and we thus see:

$$(4.4) \qquad \frac{d}{dt}T(t)x = -A\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R(\lambda;-A)xd\lambda = -AT(t)x$$

Where $x \in D_A$ and we have used the assumption that A is a closed operator. $T(t)$ thus satisfies conditions (1) and (2) of Definition 2.4. To show this is a C_0 semigroup, it remains to show that $T(t)x \to x$ as $t \to 0^+$ in the strong operator topology. By (4.4) and the Fundamental Theorem of Calculus, this is clear. We have:

$$\lim_{t\to 0+} e^{-At}x - x = \lim_{t\to 0}\int_0^t -Ae^{-A\tau}d\tau$$

Since we've already shown the integrand is bounded, this clearly tends to 0, so we do have a C_0 semigroup.

Finally, we merely have to show that $-A$ does indeed generate every $T(t)$. By definition of infinitesimal generator and (4.4), we have:

$$\frac{e^{-At}x - x}{t} = -\frac{1}{t}\int_0^t e^{-A\tau}d\tau Ax \to -Ax$$

$\square$

With the properties of analytic semigroups and some additional help from the theory of integral equations, the following theorem can be proved. See [2] for full details. We shall merely state the theorem here without proof.

Theorem 4.5 (Solution of Cauchy Problem)**.** *Given*

$$\frac{du}{dt} + A(t)u = f(t)$$

in a Banach space X with $A(t)$ a linear operator such that $u(0) = u_0$, suppose that:

(1) *D_A is dense in X and independent of t, and $A(t)$ is a closed operator.*

(2) *For each $t \in [0, t_0]$, the resolvent $R(\lambda, A(t))$ of $A(t)$ exists for all λ with Re$(\lambda) \leqslant 0$ and*

$$||R(\lambda; A(t))|| \leqslant \frac{C}{|\lambda| + 1}$$

(3) *For any t,s,τ in $[0, t_0]$,*

$$||[A(t) - A(\tau)]A^{-1}(s)|| \leqslant C|t - \tau|^{\alpha}$$

Where $\alpha \in (0,1)$ and C, α are independent of t,s,τ.

Then for any $u_0 \in X$ and for any $f(t)$ that is uniformly Hölder continuous of exponent β in $[0, t_0]$, there exists a unique solution $u(t)$ of the Cauchy problem. Furthermore, the solution is given by:

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds$$

Where $U(t,\tau)$ is a fundamental solution of $\frac{du}{dt} + A(t)u = 0$.

Remark 4.6. A fundamental solution is a generalization of Green's functions in the classical theory of ordinary differential equations.

5. Application of Semigroups to the One-Dimensional Heat Equation

As a more concrete example to show how semigroups are naturally associated to the solution of evolution equations, we consider the one-dimensional heat equation.

$$u_t = u_{xx} \tag{5.1}$$

$$u(x,0) = f$$

Where u is bounded, $t > 0$, and $x \in \mathbb{R}$. Using Fourier transforms (note that û denotes the Fourier transform of u), we obtain:

$$\frac{\partial \widehat{u}}{\partial t} + \omega^2 \widehat{u} = 0$$
$$\widehat{u}(\omega, 0) = \widehat{f}(\omega)$$

This solution is easily found to be:

$$\widehat{u} = \widehat{f} e^{-\omega^2 t}$$

Using the inversion formula for the Fourier transform, we see:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\widehat{f}e^{-\omega^2 t}d\omega \tag{5.2}$$

By the convolution theorem for Fourier transforms, we know that (5.2) is equal to the following convolution:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/4t}dy \tag{5.3}$$

Where we've used the fact that $e^{-\omega^2 t}$ is the Fourier transform of $\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$. We can now show how this is in fact an illustration of analytic semi-groups. Introduce the *heat kernel*, which is defined as:

$$K_t(x) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$$

Then, by denoting $*$ as the operation of convolution, the solution (5.3) can be rewritten as:

$$u(x,t) = K_t * f$$

Indeed, it can be shown (see [1]) that $K_s * K_t = K_{s+t}$. So, in the language of our previous sections, we see that K_t takes the place of $T(t)$ and that our arbitrary operation is in fact the operation of convolution.

6. Conclusion

In this paper we considered a rather operational approach to the solution of evolution equations. By means of a naive approach presented in section 3, we saw how we could actually give a rigorous theoretical basis to something that at first sight seemed completely absurd. A few high level results from the theory of semigroups and PDE's were presented and an illustration of the application of semigroup theory was given in the last section by means of the solution of the heat equation in one dimension.

References

[1] *Semigroups of Linear Operators*. Sheree L. Levarge.
[2] *Partial Differential Equations*. Avner Friedman. 91-130.
[3] *Semigroups of Linear Operators and Applications to Partial Differential Equations*. A. Pazy. 100-121.